Name - S.S. College, Jehanabad
Deptt - Mathematics
Topic - Adjugate and inverse of a Matrix
Class - B.Sc I (HONS)
Time - ~~10.00 P.M~~ To ~~11.00~~ 1.45 P.M
~~11.00 A.M To 11.45 A.M~~
Date - 21-09-2020
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Adjugate of a Matrix :-

Let $A = [a_{ij}]$ be a n-Square Matrix and B is a Matrix whose elements are the Co-factors of the Corresponding elements of A.

i.e $B = [A_{ij}]$ where A_{ij} = Co-factor of a_{ij} in A.

then B^T is Called the adjoint of A. or Adjugate of A and is denoted by adj A of order n.

thus if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \Rightarrow [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ \vdots & & & & \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix}$$

$$\therefore \text{adj } A = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Ex:→ Compute the Adjoint of the Matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

$A_{11} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 1+6 = 7$, $A_{12} = -\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -(-1+4) = -3$

$A_{13} = \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} = -3-2 = -5$

$A_{21} = -\begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix} = -(-2-9) = 11$ $A_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1-6 = -5$

$A_{23} = -\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = -(3+4) = -7$

$A_{31} = \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} = 4-3 = 1$ $A_{32} = -\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} = -(-2+3) = -1$

$A_{33} = \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = (1-2) = -1$

$$\therefore \text{Adj} A = \begin{bmatrix} 7 & -3 & -5 \\ 11 & -5 & -7 \\ 1 & -1 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & 11 & 1 \\ -3 & -5 & -1 \\ -5 & -7 & -1 \end{bmatrix}$$

Theorem :→ If A be a square matrix of order n, Prove that-

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| I$$

where I = unit matrix

Solution :→ Let $A = [a_{ij}]$ be a square matrix of order n

Let $B = \operatorname{adj} A = [A_{ij}]^T = [A_{ji}]_{n \times n}$

where A_{ji} = Co-factor of a_{ji} in $|A|$.

Since the matrix A and adj A both are square matrix of order n, both the product $A \cdot \operatorname{adj} A$ and $\operatorname{adj} A \cdot A$ exists and are of the order n.

clearly $a_{i1}, a_{i2}, a_{i3} \cdots a_{in}$ are elements of ith row of A and $A_{j1}, A_{j2} \cdots A_{jn}$ are the elements of jth column of adj A. Then the (i,j)th element of the product matrix $A(\operatorname{adj} A)$ will be the product of the corresponding elements of ith row of A and the jth column of adj A.

Hence the $(i,j)^{th}$ element of

$$A \cdot (\operatorname{adj} A) = a_{i1} A_{j1} + a_{i2} A_{j2} + a_{i3} A_{j3} + \cdots + a_{in} A_{jn}$$

$$= \sum_{k=1}^{n} a_{ik} A_{jk}$$

$$= 0 \quad \text{if } i \neq j$$

$$= |A| \quad \text{if } i = j$$

Hence the $(i,j)^{th}$ element of $A(\operatorname{adj} A) = |A|$ if $i = j$

$= 0$ if $i \neq j$

thus

$$A.(\text{adj}A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= |A| I$$

Similarly we can show that

$$(\text{adj}A)A = |A| I$$

$$\therefore A[\text{adj}A] = (\text{adj}A)A = |A|^n I$$

Verify the theorem $A.(\text{adj}A) = (\text{adj}A).A = |A| I_3$

When $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and also $[\text{adj}A|.|A| =$

Solution :→ We have

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (1-0) + 2(4-3) = 1 + 2 = 3$$

$A_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$ $\quad A_{12} = -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2$ $\quad A_{13} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4-3 = 1$ 5

$A_{21} = \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = +4$ $\quad A_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5$ $\quad A_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -2$

$A_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2$ $\quad A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 4$ $\quad A_{33} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$

$$\therefore \text{Adj}A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -5 & -2 \\ -2 & 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A(\text{Adj}A) = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+2 & 4+0-4 & -2+0+2 \\ 2-2+6 & 8-5+0 & -4+4+0 \\ 3-4+1 & 12-10-2 & -6+8+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3 \cdot I_3$$

$$= |A|\, I_3$$

$|I_3$

$|A| = |A|^n$

Similarly $(\text{Adj}A)A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}_{3\times 3}$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3 I_3 = |A|\, I_3$$

Now it remains to show

$$|\operatorname{adj} A| \cdot |A| = |A|^n$$

Now $\operatorname{adj} A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$

$$|\operatorname{adj} A| = \begin{vmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -5 & 4 \\ -2 & 1 \end{vmatrix} - 4 \begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} + (-2) \begin{vmatrix} -2 & -5 \\ 1 & -2 \end{vmatrix}$$

$$= (-5+8) - 4(-2-4) - 2(4+5)$$

$$= 3 + 24 - 18$$

$$= 9$$

$$|A| = 3$$

$$\therefore |\operatorname{adj} A||A| = 9 \times 3 = 27$$

$$|A|^3 = 27$$

thus $|\operatorname{adj} A|\,|A| = |A|^3$

thus it is verified that

$$|\operatorname{adj} A||A| = |A|^n$$